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# Stable localized vortex solitons

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We demonstrate that parametric interaction of a fundamental beam with its second harmonic in bulk media, in the presence of self-defocusing third-order nonlinearity, gives rise to the first ever examples of *completely stable* localized ring-shaped solitons with intrinsic vorticity $n=1$ and $n=2$. The stability is demonstrated both in direct simulations and by computing eigenvalues of the corresponding linearized equations. A potential application of the (2+1)-dimensional ring solitons in optics is a possibility to design a reconfigurable multi-channel system guiding signal beams.

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Extended and localized vortices are fundamental structures in nature. Well-known examples are vortices in liquid helium, Abrikosov vortices in superconductors [1], Rossby-wave vortices in geophysical hydrodynamics [2], and vortices in Bose-Einstein condensates [3]. Since the pioneering experimental results reported in Ref. [4] *optical* vortices have been of great interest too.

Besides their importance as fundamental objects, optical vortex solitons may find use in all-optical information processing applications. Conventional vortices require an infinitely extended nonzero background (therefore, we refer to them below as *delocalized* vortices), which in practice means that they can be observed in beams with a large cross-section area [5]. A different class of dynamical objects are (2+1)-dimensional *localized* optical vortex solitons (or *rings*, as we call them below), i.e., finite-size self-guided beams with an internal vorticity. Due to the presence of the vorticity, the beam's cross section has an annular shape, with a hole inside. However, theoretical [6] and experimental [7] studies have shown that, in materials with purely quadratic or saturable nonlinearity, ring solitons are strongly unstable against azimuthal perturbations. Materials with a mixed nonlinear response may prove to be more productive. It was argued [8] that rings are stable against both small perturbations and collisions between them in optical media with a cubic-quintic nonlinear response. However, longer simulations demonstrate that these ring vortices are also subject to a weak instability against azimuthal perturbations in both (2+1)D- and (3+1)D geometries [9], which eventually breaks the ring into a set of zero-spin solitons flying out in tangential directions.

Thus, there is a challenging question: Can *truly stable* ring solitons exist in models with a realistic nonlinearity? In this work, we propose a model with a mixed $\chi^{(2)}$ (quadratic)-$\chi^{(3)}$ (cubic) nonlinearity which indeed supports stable ring solitons. We prove the stability of rings rigorously, i.e., by both direct simulations (including collisions) and the full-scale linear-stability analysis. This yields the first example of stable vortex rings in optics. (Note that the only other known species of stable localized vortices occur in hydrodynamics and superconductivity, i.e., very far from optics.)

Besides its fundamental importance, this result may also have direct applications. Indeed, the most promising concept in photonics is ''light guided by light.'' In particular, a lot of attention was given to an idea according to which the conventional (delocalized) optical vortex can be used as a guide for a weak signal beam [10]. Unlike this single-channel scheme, the existence of stable vortex rings makes it possible to design, inside a sample of the same size, an easily changeable *multichannel* configuration.

Following the derivation procedure of Ref. [11], we arrive at a system of equations for the fundamental- and second-harmonic (FH and SH) fields $u$ and $w$,

$$
\begin{gathered}
iu_z+\nabla^2 u-\beta u+u^*w-(|u|^2/4+2|w|^2)u=0,\\
2iw_z+\nabla^2 w-\alpha w+u^2/2-(4|w|^2+2|u|^2)w=0,
\end{gathered}
\tag{1}
$$

where $\nabla^2\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$ is the transverse diffraction operator, $\alpha\equiv 2(\Delta+2\beta)$, $\beta$ is a nonlinear shift of the fundamental-harmonic's propagation constant, and $\Delta$ is the wave-vector mismatch between harmonics. In this work, we consider the case when the $\chi^{(3)}$ nonlinearity is self-defocusing.

Stationary localized solutions with vorticity (''spin'') $n$ have the form $u=U(r)\exp(in\phi)$ and $w=W(r)\exp(2in\phi)$, where $r$ and $\phi$ are the polar coordinates in the transverse plane, and real functions $U(r)$ and $W(r)$ are to be found as localized solutions of the system

$$
\begin{gathered}
\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}-\frac{n^2U}{r^2}-\beta U+UW-F_uU=0,\\
\frac{d^2W}{dr^2}+\frac{1}{r}\frac{dW}{dr}-\frac{4n^2W}{r^2}-\alpha W+\frac{U^2}{2}-F_wW=0,
\end{gathered}
\tag{2}
$$

where $F_u\equiv(U^2/4+2W^2)$ and $F_w\equiv(4W^2+2U^2)$. We solved Eqs. (2) by means of the relaxation technique, finding domains of existence of localized solutions for different $n$, typical examples of which are shown in Fig. 1. When $\beta$ is small (a low-power regime), ring solitons are narrow. The beam's amplitude at first increases with $\beta$ and then saturates, while the ring's width keeps increasing because of the self-defocusing effect of the $\chi^{(3)}$ term. This is similar to what is

 

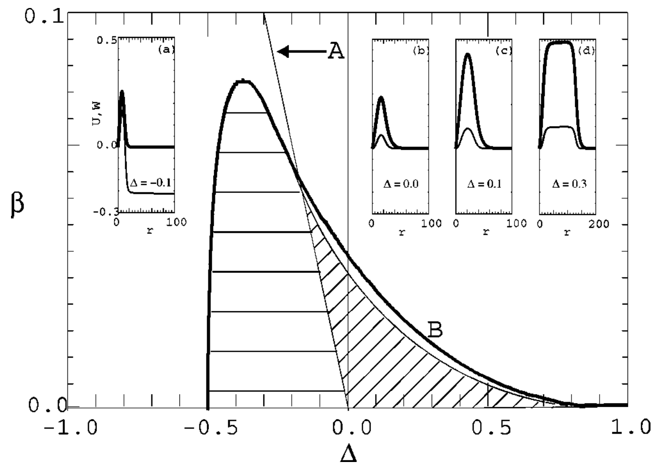


FIG. 1. Existence regions for different solutions of Eqs. (1) in the $(\Delta,\beta)$ plane. Insets show examples of a *semilocalized* vortex, supported by a finite background in the second harmonic (a), and localized rings (b,c,d) for $\beta=0.02$ and different values of $\Delta$, with thick and thin lines showing the first- and second-harmonic radial functions $U(r)$ and $W(r)$, respectively. Delocalized vortices exist everywhere below curve *B*. The meaning of line *A* is explained in the text. The existence domain for rings and zero-spin bright solitons is diagonally hashed. The domain of existence of semilocalized vortices, see an example in (a), is horizontally hashed. (Note that delocalized and semilocalized vortices, in contrast to the localized rings and zero-spin bright solitons, may exist also at $\beta<0$.)

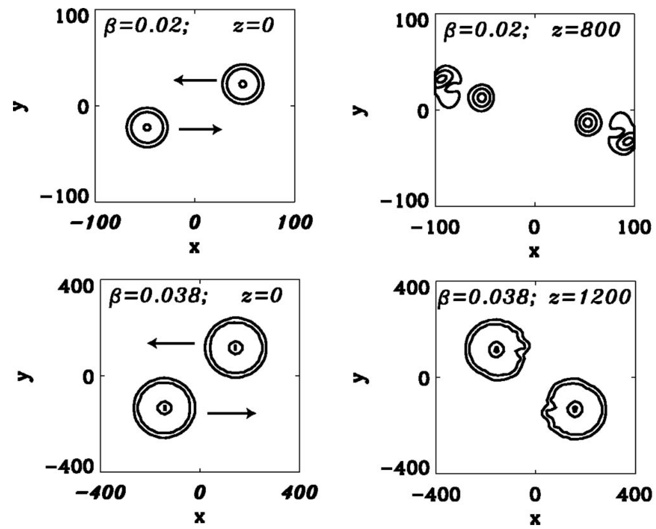


FIG. 2. Examples of collisions between unstable (upper) and stable (lower) ring solitons of Eqs. (1) at $\Delta=0.1$. Usually, an unstable ring splits into two zero-spin bright solitons. Rings which survive the collisions are broad enough.

known for the cubic-quintic model with focusing $\chi^{(3)}$ and defocusing $\chi^{(5)}$ terms [8,9]. The similarity is not surprising, as in the *cascading limit* of large $\Delta$ the SH component can be eliminated from Eqs. (2), $w\approx u^2/(4\Delta)$, and the remaining FH equation reduces to the cubic-quintic model.

The *existence* results for various types of solitons of Eqs. (1) are summarized as a diagram in the parametric plane $(\Delta,\beta)$, see Fig. 1. A noteworthy feature is coexistence of all the types of solutions, i.e., conventional delocalized vortices, semilocalized vortices (localized in the *u*-component and delocalized in the *w* component [12]), and fully localized rings, along with zero-spin bright solitons (the latter were considered in Ref. [13].)

Localized rings bifurcate from delocalized vortices along the curve $\beta_{\rm cr}(\Delta)$ in $(\beta,\Delta)$-parameter space. This curve is the upper boundary of the diagonally hashed region in Fig. 1. At $\beta>0$, semilocalized vortices are a continuation of the rings: these two types of solutions merge when the background in the SH component vanishes. It follows immediately from the second equation (2) that this happens at $\Delta=-2\beta$, on the line *A* in Fig. 1. *All* the semilocalized vortices with spin $n=1$ are known to be unstable [12]. It seems very plausible that they are unstable too for any $n>1$. Below, we concentrate on solutions in the form of localized rings, and their stability. In fact, the stability of multidimensional solitons is a very complex issue. We address it first by simulating the propagation of azimuthally perturbed ring solitons and their collisions. A conclusion strongly suggested by the simulations is that the rings which are close enough to the upper boundary of their existence domain (i.e., broad flat-top rings) are stable, whereas narrower ones are unstable against azimuthal perturbations. This conclusion is consistent with recent results obtained for the cubic-quintic model, where it was clearly shown that making the ring broader strongly, but *not* completely, suppressed its azimuthal instability [9]. The stabilization is related to the fact that the broad-ring solutions are close to modulationally *stable* plane waves. However, this argument only shows a general trend and cannot predict if azimuthal instabilities are completely eliminated. Therefore, a more detailed analysis is necessary to search for genuine stable rings in the model.

Our simulations show that relatively narrow unstable rings with $n=1$ break up into two zero-spin filaments (bright solitons), which is also a generic outcome of the development of the azimuthal instability in the saturable and cubic-quintic models [6,9]. In contrast with this, broad rings not only remain completely stable in the course of very long propagation, but also survive collisions between themselves; see Fig. 2.

Direct simulations may be sufficient to predict experimental observation of stable rings, but the principle issue of stability can only be resolved by comprehensive analysis of small perturbations around the ring vortices. To this end, we add infinitesimal complex perturbations $\epsilon(z,r,\theta)$ to stationary solutions of Eqs. (1) and (2) with the vorticity *n*,

$$u=[U(r)+\epsilon_1(z,r,\theta)]e^{in\theta},$$
$$w=[W(r)+\epsilon_2(z,r,\theta)]e^{2in\theta}. \tag{3}$$

A general perturbation $\epsilon_m(z,r,\theta)$ may always be expanded into a series, each term of which has its own vorticity *J*,

$$\epsilon_m=\sum_J\,[\xi^+_{Jm}(r)e^{i(\lambda z+J\theta)}+\xi^-_{Jm}(r)e^{-i(\lambda^* z+J\theta)}], \tag{4}$$

where $\lambda$ is a stability eigenvalue, $\lambda^*$ being its complex conjugate. Substituting this into Eqs. (1) and linearizing, we arrive at a non-self-adjoint eigenvalue problem,

$$\lambda \vec{\xi}_J = \begin{bmatrix} A_+ & B & C & D \\ -B & -A_- & -D & -C \\ C/2 & D/2 & E_+/2 & F/2 \\ -D/2 & -C/2 & -F/2 & -E_-/2 \end{bmatrix} \vec{\xi}_J, \tag{5}$$

where $\hat{\xi}_J \equiv (\xi^+_{J\,1}, \xi^-_{J\,1}, \xi^+_{J\,2}, \xi^-_{J\,2})$, $A_\pm = \hat{L}^\pm_{J\,1} - (U^2/2 + 2W^2)$, $B = W - U^2/4$, $C = U - 2UW$, $D = -2UW$, $E_\pm = \hat{L}^\pm_{J\,2} - 2(4W^2 + U^2)$, $F = -4W^2$, and

$$\hat{L}^\pm_{J\,1} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}(n \pm J)^2 - \beta,$$

$$\hat{L}^\pm_{J\,2} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}(2n \pm J)^2 - (4\beta + 2\Delta). \tag{6}$$

Instability is accounted for by eigenvalues with $\mathrm{Im}\,\lambda \neq 0$ (the present system being Hamiltonian, eigenvalues appear in complex conjugate pairs or quadruplets). The problem's continuous spectrum consists of real intervals $\Gamma \leqslant \lambda < \infty$ and $-\infty < \lambda \leqslant -\Gamma$, with $\Gamma \equiv \min(\beta, 4\beta + 2\Delta)$.

To analyze eigenvalue problem (5), we replaced the differential operators by their fifth-order finite-difference approximations and solved the resulting algebraic eigenvalue problem numerically. We mostly used grids with 400 to 800 points, but up to 1200 points were used in regions where a change of the stability occurs. To verify the precision of the numerical code, we also used another technique, based on the relaxation method for solving two-point boundary-value problems. Although limited to finding real eigenvalues, the latter method admits a high degree of precision control without much of the computational overhead of other methods. For instance, it has been recently used to a great effect in finding a small stability window for higher-order solitons in a third-harmonic-generation model, which would have otherwise been overlooked [14]. A comparison between the spectral and relaxation methods has shown that the former has a good precision for the number of grid points we were using: a numerical error in calculating the stability-boundary values of $\beta$ is estimated to be $\delta\beta_{\mathrm{st}}(\Delta) \sim 10^{-5}$ for 1200 grid points.

Results of the linear stability analysis for the *fundamental* rings ($n=1$) are summarized in Figs. 3 and 4. We considered the perturbations with $J=0,\ldots,\pm 5$, and have found that instability may be generated by $J=\pm 2$ and $\pm 3$. The strongest instability corresponds to $J=\pm 2$, eventually leading to breakup of the ring into two zero-spin solitons. In all the cases considered, we have found that there is a stability-change value $\beta_{\mathrm{st}}$, at which the largest instability eigenvalue $\mathrm{Im}\,\lambda$ vanishes, and remains, along with all the other ones, exactly (up to the numerical accuracy) equal to zero in the *stability window*, $\beta_{\mathrm{st}} < \beta < \beta_{\mathrm{cr}}$ [recall $\beta_{\mathrm{cr}}(\Delta)$ is the upper existence boundary of the ring-soliton family]; in other words, thin rings are unstable and broad ones are stable, in accord with results of direct simulations presented above. The existence of the window is clearly illustrated by Fig. 3. For the fundamental rings, the stability window occupies up to $\approx 8\%$ of the existence domain $[0, \beta_{\mathrm{cr}}(\Delta)]$. In the direct simulations, the ring solitons belonging to the window propagate indefinitely long without any visible instability, surviving various perturbations and collisions with other solitons.

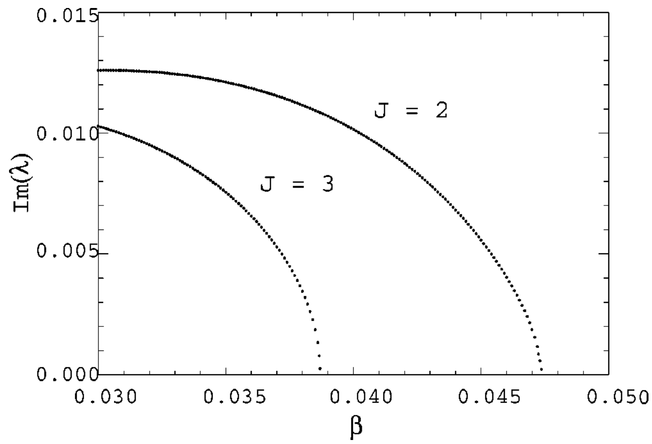


FIG. 3. Unstable eigenvalues for $\Delta=0$, corresponding to $J=\pm 2$ and $J=\pm 3$ (only Im $\lambda$ is shown). $J=\pm 2$ generate the strongest instability, which is, however, no longer present at $\beta \geqslant 0.0475$, while the rings exist up to $\beta \approx 0.0518$; thus, the rings are *completely stable* between these two values of the propagation constant. Within the numerical accuracy, Im $\lambda=0$ for $J=(\pm 1, \pm 4, \pm 5)$, therefore these eigenvalues are not shown.

In addition to the fundamental rings, we have also studied the linear spectrum of higher-order rings with $n=2$ and 3. The rings with $n=3$ were found to demonstrate some persistent weak instability associated with $J=\pm 1$ perturbation, but the rings with $n=2$ have their own stability window, occupying up to $\approx 5\%$ of the existence domain. We stress that, even for $n=2$, the width of the window exceeds a possible numerical error by two orders of magnitude. It is necessary to point out that, while stable localized vortices with

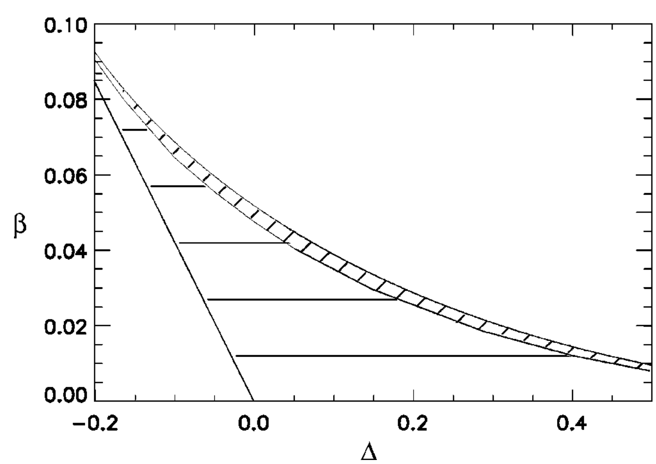


FIG. 4. Areas of existence and stability for fundamental ($n=1$) rings in Eqs. (1). Regions of stable and unstable rings are hashed diagonally and horizontally, respectively.

$n=1$ were earlier known in superconductivity and hydrodynamics (but not in optics), *no example* of stable $n=2$ vortex ring has been known before [1,2]. Thus, our model provides for the first example of a stable double-charge localized vortex.

Note that second-order (double-humped) solitons have been found to be stable in several (1+1)D models [15], and such solitons were observed in recent experiments [16]. Due to the shape of their cross section, (2+1)D rings and zero-spin single-humped solitons are analogous, respectively, to the double-humped and fundamental solitons in (1+1)D. Thus, the coexistence of the stable solitons with zero and nonzero spins resembles the coexistence between stable single- and double-humped (1+1)D solitons. However, a principal difference is that the stable (2+1)D solitons with zero and nonzero spins coexist while belonging to different *topological classes*.

Finally, we address a possibility of experimental realization of optical media necessary for observation of stable rings. Although no conventional nonlinear material with strong $\chi^{(2)}$ nonlinearity directly satisfies our requirement to have a negative $\chi^{(3)}$ coefficient for both the FH and SH frequencies, there are two possibilities to achieve this purpose: (i) by creating a layered medium in which some layers provide for the $\chi^{(2)}$ nonlinearity, and others for the self-defocusing Kerr nonlinearity, and (ii) by engineering special $\chi^{(2)}$ QPM gratings [17]. In the latter case, induced cubic nonlinearity may be equal in strength to the intrinsic $\chi^{(2)}$ nonlinearity. We checked the former possibility by numerical simulations and have found it to work well (details will be presented elsewhere). Thus, media that may support stable vortex rings are within the reach of the modern-day experiment. Last, ring solitons studied in this work may also be expected in interacting molecular and atomic Bose-Einstein condensates, which are described by a model similar to ours [18].

In conclusion, we have analyzed the existence and stability of (2+1)-dimensional ring solitons with intrinsic vorticity in optical media with competing quadratic and self-defocusing cubic nonlinearities. Sufficiently broad ring solitons with the spin $n=1$ and 2 have been shown to be stable, both in direct dynamical simulations and analyzing eigenvalues of the linearized equations. Further stable topological solitons have been shown to coexist with zero-spin bright solitons in the competing nonlinearities model. A potential application of the (2+1)D ring solitons is a possibility to design a reconfigurable multichannel system guiding signal beams. The results obtained in this work may be readily applied to models of different physical content, including Bose-Einstein condensate applications.

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